Worksheet Five **MATH 104** Name:____________

WORK ON THIS ASSIGNMENT IN GROUP OF 2-4. TURN IN YOUR WORK INDIVIDUALLY IN CLASS. YOU CAN USE YOUR NOTES FOR THIS ASSIGNMENT.

3.7: Rational Functions

Horizontal Asymptotes of Rational Functions

- A rational function is a quotient of two polynomial functions. We represent a rational function in the following form.
  $r(x) = \dfrac{a_n x^n + ... + a_0}{b_m x^m + ... + b_0}$.
- Let r be a rational function in the most simplified form (no common factor between the numerator and the denominator) and $c_1 ... c_k$ be zeros of the denominator, then $x = c_1 ... x = c_k$ are the **vertical** asymptotes of the function.
- If a factor $(x - a)$ is a factor of both numerator and the denominator, then graph of r either has a hole or a vertical asymptote at $x = a$. (For example, $f(x) = \dfrac{x^2 - 9}{x - 3}$ has no vertical asymptote but a hole at $x = 3$; $f(x) = \dfrac{x^2 - 9}{(x-3)^2}$ has a vertical asymptote at $x = 3$.)
- Let r be a rational function in the most simplified form (no common factor between the numerator and the denominator) and $d_1 ... d_j$ be zeros of the numerator, then $x = d_1$, $x = d_2$, ..., $x = d_j$ are the x-intercepts (the points $(d_1, 0), (d_2, 0), \cdots, (d_j, 0)$).
- The y-intercept of any function $y = r(x)$ is $r(0)$ (the point $(0, r(0))$).

  The horizontal asymptote of a rational function can be determined by looking at the degrees of the numerator and the denominator.

  - The *degree of numerator is less than the degree of denominator: horizontal asymptote at* $y = 0$.
  - *The degree of numerator is greater than the degree of denominator by one:* **no horizontal asymptote***; slant asymptote. Use long division to find the slant asymptote.*
  - *The degree of the numerator is equal to the degree of the denominator: horizontal asymptote at ratio of leading coefficients.*
  - *The degree of numerator is* **greater** *than the degree of the denominator by* **more than one***: The end behavior is not linear.*

- In other words, for $r(x) = \dfrac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0}$,

  1. If $n < m$, $y = 0$ is a horizontal asymptote.
  2. If $n > m$ then the graph has no horizontal asymptote. If $n = m + 1$ then it has **slant** asymptote. That is the quotient rendered by dividing the numerator by the denominator. $r(x) = ax + b + \dfrac{R(x)}{D(x)}$ implies $y = ax + b$ is the slant asymptote. Since $\dfrac{R(x)}{D(x)} \to 0 \;\; as \;\; x \to \pm\infty$
  3. If $n = m$ then $y = \dfrac{a_n}{b_m}$ is the horizontal asymptote.

Graphing

- Find x-intercept, y-intercept and asymptotes of the function.
- Divide the x-axis into pieces that are divided by x-intercepts and vertical asymptotes. Find test point/s in each piece (y-intercept can count as one test point). Remember by intermediate value theorem, if the graph is not broken and does not cross x-axis within each piece, then the sign of the function does not change.
- Rational functions go to infinity at either side of a vertical asymptote. Use the test points to choose between $\pm\infty$ of functions around vertical asymptotes.
- Use the test points and horizontal asymptotes to find the behavior as $x \to \pm\infty$ and graph.
- Graph. The local minimum/maximum and more accurate graphing is part of Calculus.

Simplifying the Sum of Rational Expressions

- Make sure each expression is simplified. (Assuming that both original and the simplified form are in the same domain.)
- Find the least common denominator. This is going to be the new denominator.
- Multiply all rational terms to make the new numerator.
- After forming the new fraction, check if it can be simplified again.

Note: The asymptotic behavior of a graph is important in Calculus, Science and Engineering. We look at the asymptotic behavior of $\overbrace{\text{rational functions}}^{\text{Section 3.7}}$, $\overbrace{\text{exponential functions}}^{\text{Chapter 4}}$, $\overbrace{\text{logarithmic functions}}^{\text{Chapter 4}}$, $\overbrace{\text{some trigonometric functions}}^{\text{Chapter 6}}$, and $\overbrace{\text{some inverse trigonometric functions}}^{\text{Chapter 6}}$. Please be aware of them and learn them as we go.

1. Find the asymptotes of the following functions.

(a) $f(x) = \dfrac{x+1}{5x^2+1}$

(b) $g(x) = \dfrac{5x^2-1}{3x^2+1}$

(c) $h(x) = \dfrac{x^2-2x-1}{x-1}$

2. Let $f(x) = \dfrac{x^2+9}{144-9x^2}$.

(a) Find all vertical asymptotes of the graph of $y = f(x)$.

(b) Find all horizontal asymptotes of the graph of $y = f(x)$.

3. If $\frac{1}{x} - \frac{1}{x-2} = \frac{x-10}{6x}$, then the solution set is

(A) $\{-4, -8\}$ (C) $\{-4, 8\}$ (E) $\{4, -4\}$ (G) $\{4\}$

(B) $\{4, 8\}$ (D) $\{4, -8\}$ (F) $\{8, -8\}$ (H) No Solution

4. $\dfrac{\frac{1}{x} + \frac{1}{x^4}}{\frac{1}{x}}$ is equivalent to

(A) 1 (B) $\frac{1}{x^4}$ (C) $\frac{x+1}{x}$ (D) $\frac{x^2+1}{x^2}$ (E) $\frac{x^3+1}{x^3}$ (F) $\frac{x^4+1}{x^4}$

5. $\frac{5}{x-1} - \frac{3}{x-2}$ is equivalent to

(A) $\frac{2}{(x-1)(x-2)}$ (B) $\frac{2x-16}{(x-1)(x-2)}$ (C) $\frac{2x-11}{(x-1)(x-2)}$ (D) $\frac{2x-7}{(x-1)(x-2)}$ (E) $\frac{2x-4}{(x-1)(x-2)}$

6. Let $f(x) = \dfrac{x+b}{x-b}$ where $b > 0$.

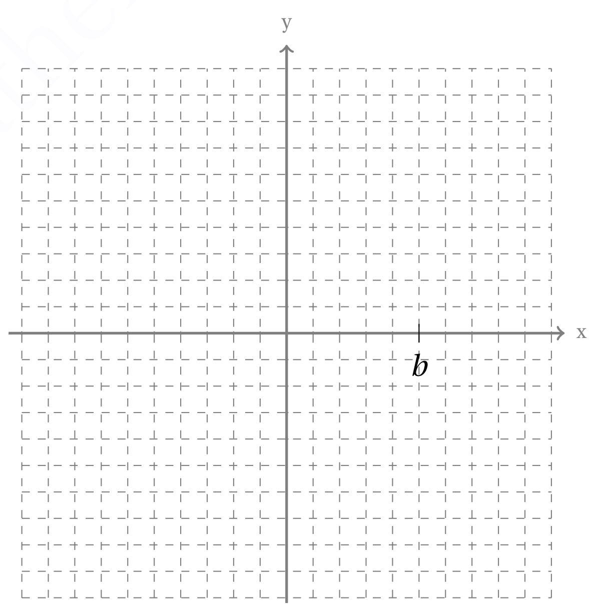
(a) Find all vertical asymptotes of the graph of $y = f(x)$ in terms of b.

(b) Find all horizontal asymptotes of the graph of $y = f(x)$.

(c) What is the x-intercept of $y = f(x)$ in terms of b?

(d) What is the y-intercept of $y = f(x)$?

(e) Graph $y = f(x)$



7. Consider $f(x) = \dfrac{x^2 - 11x + 28}{x^2 - 4}$.

(a) Find x-intercepts of the graph of f.

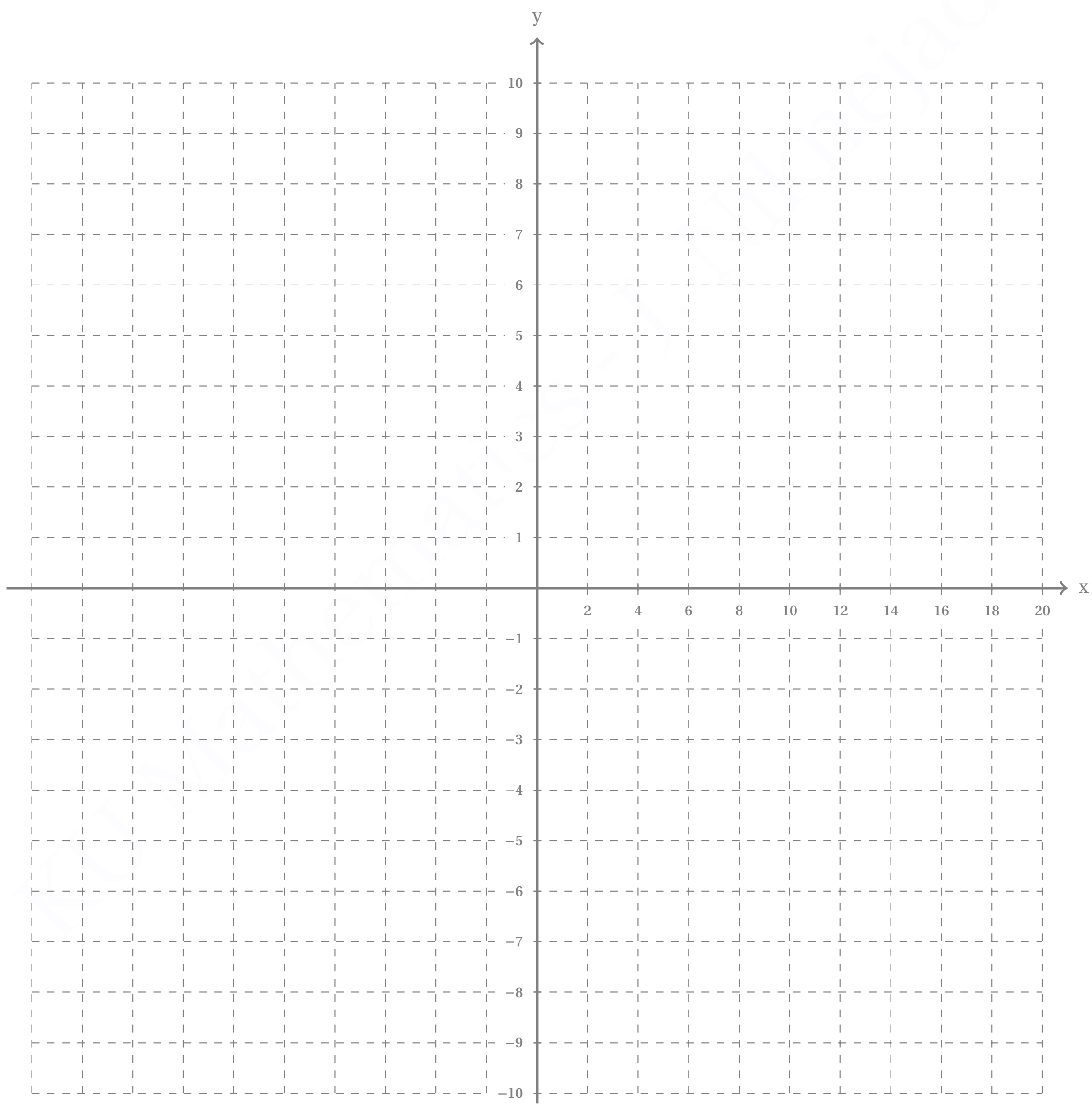
(b) Find y-intercept of the graph of f.

(c) Find all asymptotes of the graph. Label as vertical, horizontal or oblique asymptote.

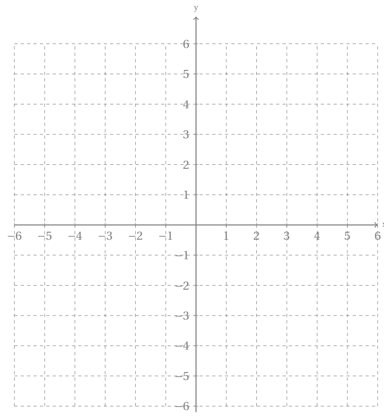
(d) Compute the following Values:

$$\begin{aligned} f(-10) &\approx \\ f(5.5) &\approx \\ f(10) &\approx \end{aligned}$$

(e) Graph f.



8. Graph $r(x) = \dfrac{x^2 - 16}{x - 4}$. (Hint: Can you simplify $r(x)$?)



9. Find the average rate of change in $f(x) = \dfrac{1}{x}$ over the interval $[b, b+h]$; simplify as much as possible.

10. Simplify, within the domain, as much as possible $\dfrac{(x^2+1)(x-1)^2}{x^4-1}$. Watch Gateway Video 96.

11. Simplify, within the domain, as much as possible $\dfrac{xy+3zy}{x^2+6xz+9z^2}$. Watch Gateway Video 97.

12. Solve $25\left(\dfrac{g}{g+1}\right)^2 - 10\left(\dfrac{g}{g+1}\right) + 1 = 0$ for g. Watch Gateway Video 53.

INDIVIDUAL WORK

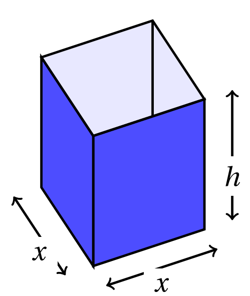
UPLOAD TO CANVAS OR SUBMIT IN CLASS BEFORE DUE DATE. DISCUSSING THESE QUESTIONS IN YOUR GROUP IS ENCOURAGED BUT MAKE SURE YOU ARE TURNING IN YOUR OWN WORK.

13. *(3 points)* Find the average rate of change in $f(x) = \dfrac{3}{x+9}$ over the interval $[b, b+h]$; simplify as much as possible.

14. We are building a rectangular box with **no lid** with height h units and a square base with dimension x units. The only restriction is that the volume of the box has to be 45 unit^3.

    (a) *(1 point)* Express h as a function of x.

    (b) *(1 point)* Express the surface area as a function of x.



15. Consider $f(x) = \dfrac{x^2 - 11x + 28}{4 - x^2}$.

(a) *(0.5 points)* Find x-intercepts of the graph of f.

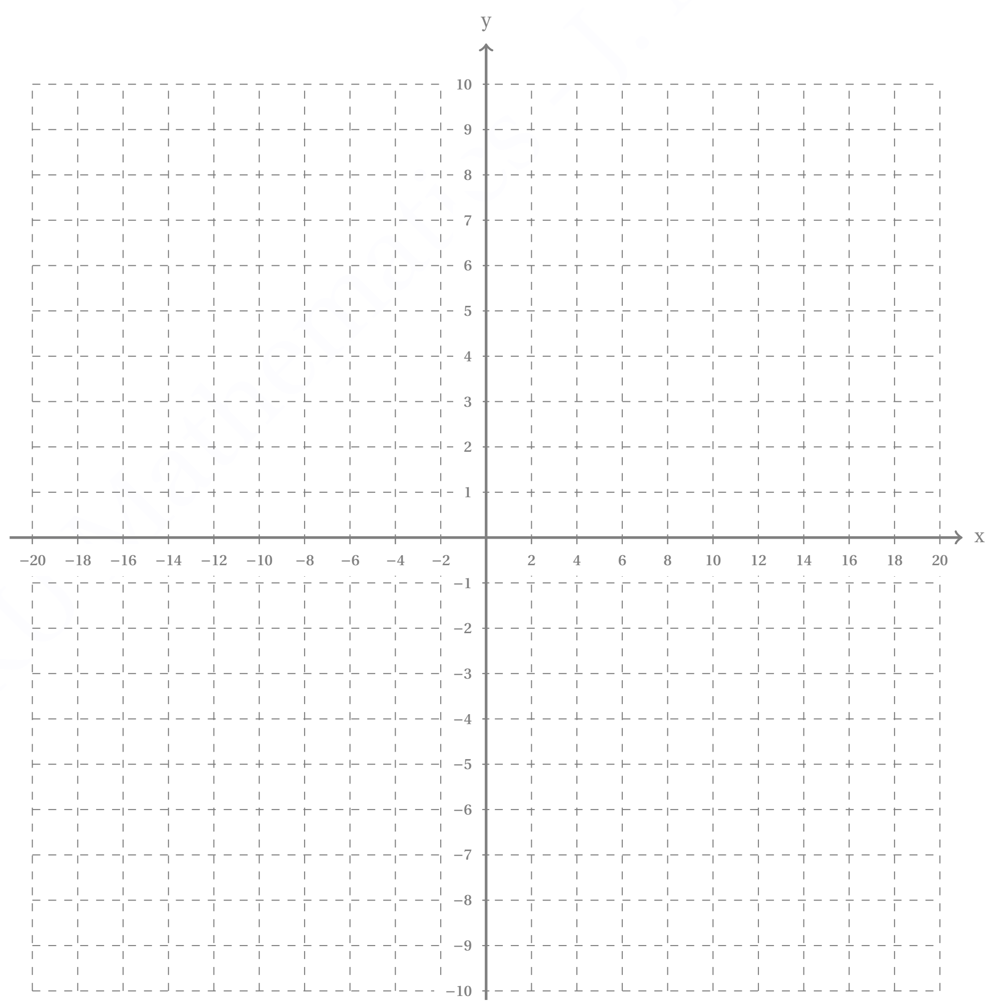
(b) *(0.25 points)* Find y-intercept of the graph of f.

(c) *(0.75 points)* Find all asymptotes of the graph. Label as vertical, horizontal or oblique asymptote.

(d) *(0.75 points)* Find the following values

$$\begin{aligned} f(-10) &\approx \\ f(5.5) &\approx \\ f(10) &\approx \end{aligned}$$

(e) *(0.75 points)* Graph f.



Related Videos

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2. Rational Graphing: https://mediahub.ku.edu/media/t/1_cx08u0he
3. Watch Gateway Video 49: https://mediahub.ku.edu/media/MATH+104+-+049.m4v/0_wp1m5hon
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